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AN INSTRUMENT FOR MEASURING SPOT SIZE

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Richard B. Kuhn
and
Daniel Levine

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SECTION I - INTRODUCTION

The determination of the spot size of a cathode-ray tube has been, at best, a subjective process. Thus, the shrinking-raster method of measuring spot sizes relies on the judgment of the observer to determine when a set of lines appears to merge. Similarly, any attempt to measure the spot diameter with a microscope is also dependent upon the observer because of the way the flux density varies over the area of the spot.[1]

This report describes an instrument for determining spot size; the instrument is independent of the observer. The spot size is defined[2] as being equal to the side of a square within which the current density in the Gaussian distribution about a stationary beam $I_c e^{-k^2 r^2}$ may be redistributed uniformly with current density I_c.

Then

$$k = \frac{\sqrt{\pi}}{w} = \frac{1.7725}{w}. \tag{1}$$

The equipment may be designed so the measured value of spot size has an rms uncertainty of less than four percent.

[1] Soller, T.; Starr, M. A.; and Valley, G. E., Jr.: Cathode-Ray Tube Displays. New York, McGraw-Hill Book Co., Inc., 1948. pp 590-600.

[2] Levine, Daniel: Significance of Line Width in PPI Displays. GER-6142. Akron, Ohio, Goodyear Aircraft Corporation. 16 June 1954; p 37, Equation D-1.

SECTION II - DESCRIPTION

The instrument (see Figure 1) consists of a chart, a lens to focus the crt spot onto the chart, and a lens to focus the light passing through the chart onto a phototube. A filter is placed in the light beam to pass only the short persistent component of the phosphor light radiation. In practice, the filter would undoubtedly be mounted on the object lens; however, its position is unimportant.

The chart is a series of opaque spaces separated by clear apertures of various widths (see Figure 2). The opaque spaces and the first clear aperture have a width (W) at least four times the width of the largest spot to be measured. The width of the narrowest clear space is approximately four-fifths of the size of the smallest spot to be measured.

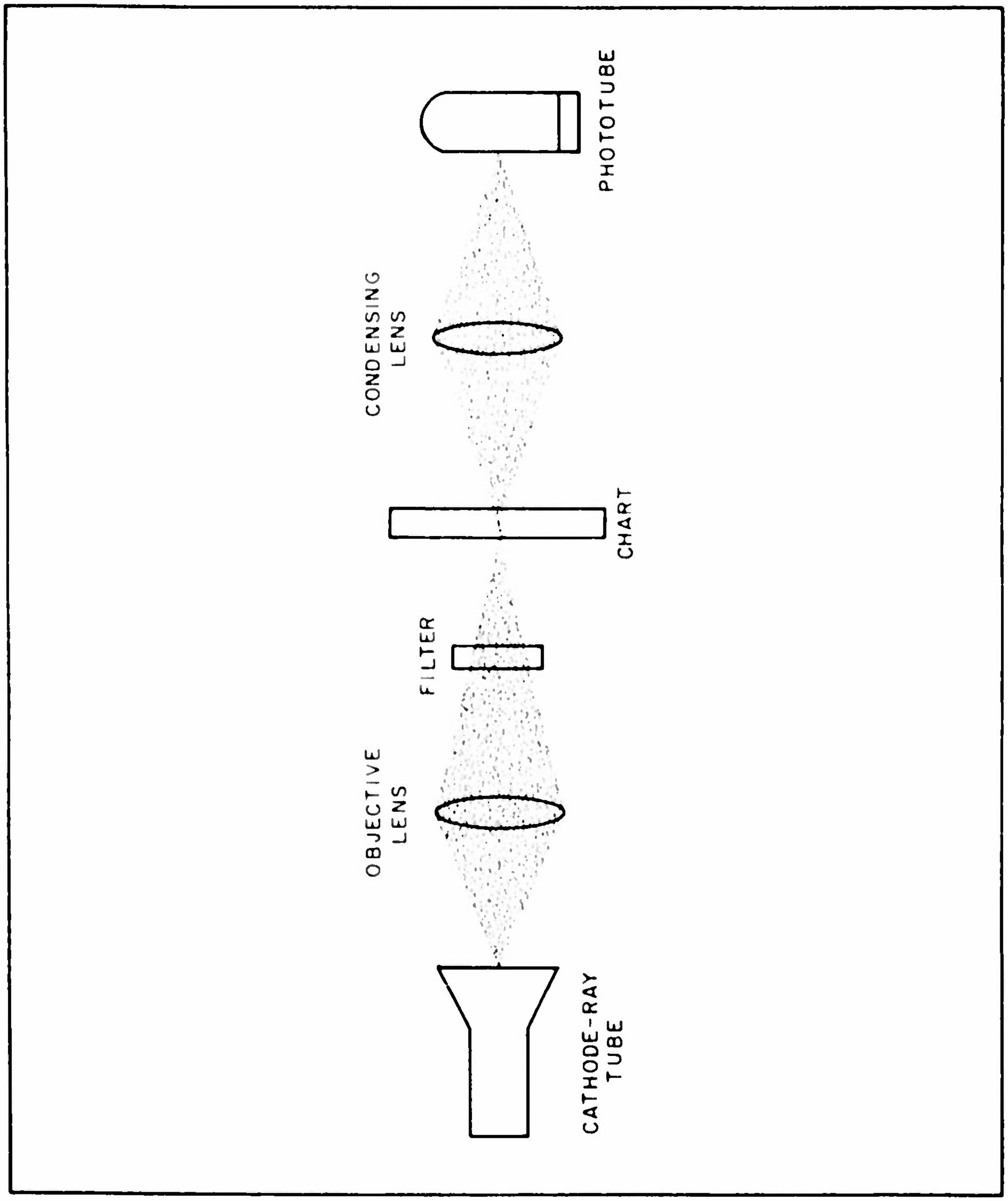

Figure 1 - Representation of Spot Size Measuring Instrument

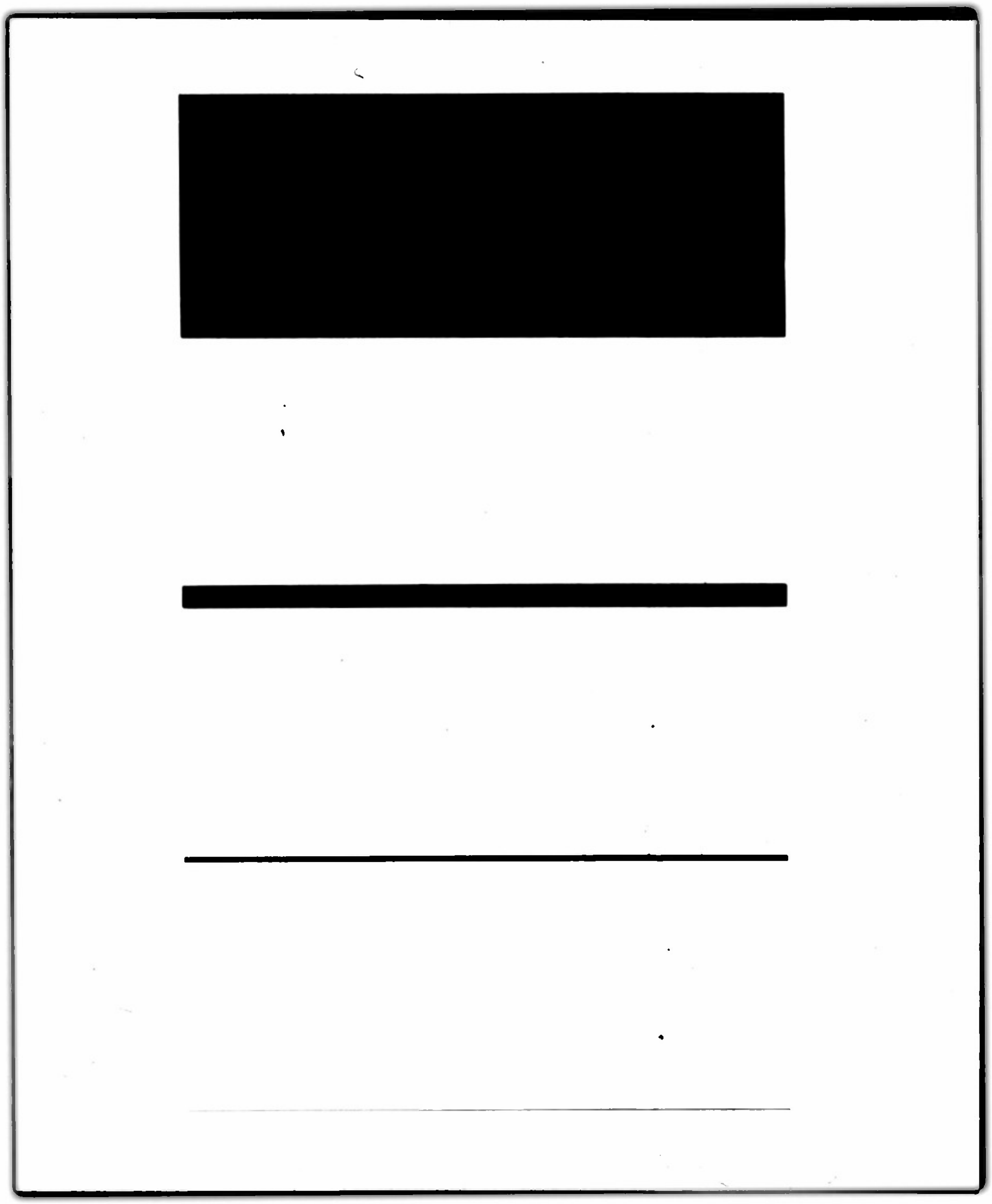

Figure 2 - Chart for Spot Size Measurement

SECTION III - THEORY

When a spot is focused within a single clear aperture, it has been shown that, if the phosphor decay is neglected, the relative light output is

$$\frac{L}{L_o} = \frac{1}{2}\left\{H\left[k(x + W)\right] - H(kx)\right\}, \tag{2}$$

where x is the distance measured from the center of the beam to the leading edge of the aperture.

This has its maximum value when

$$x = -\frac{W}{2}; \tag{3}$$

i.e., when the spot is centered in the aperture.

If the maximum is denoted by subscript M, then

$$\frac{L_M}{L_o} = \frac{1}{2}\left[H\left(\frac{kW}{2}\right) - H\left(-\frac{kW}{2}\right)\right] = H\left(\frac{kW}{2}\right). \tag{4}$$

Substitution of the definition of k into Equation 4 yields

$$\frac{L_M}{L_o} = H\left(\frac{\sqrt{\pi}}{2}\,\frac{W}{w}\right). \tag{5}$$

The light ratio may be measured by placing the output of the phototube on a suitable recorder. The light through a chart (see Figure 3A) as recorded at the phototube output is illustrated in Figure 3B for the case where the spot size is equal to the width of the second aperture.

When the spot is in the center of the first aperture there is no significant limiting of the light output, and the value of L_o is determined (see Figure 3C, left). However, when the spot is at the center of an aperture that is equal to

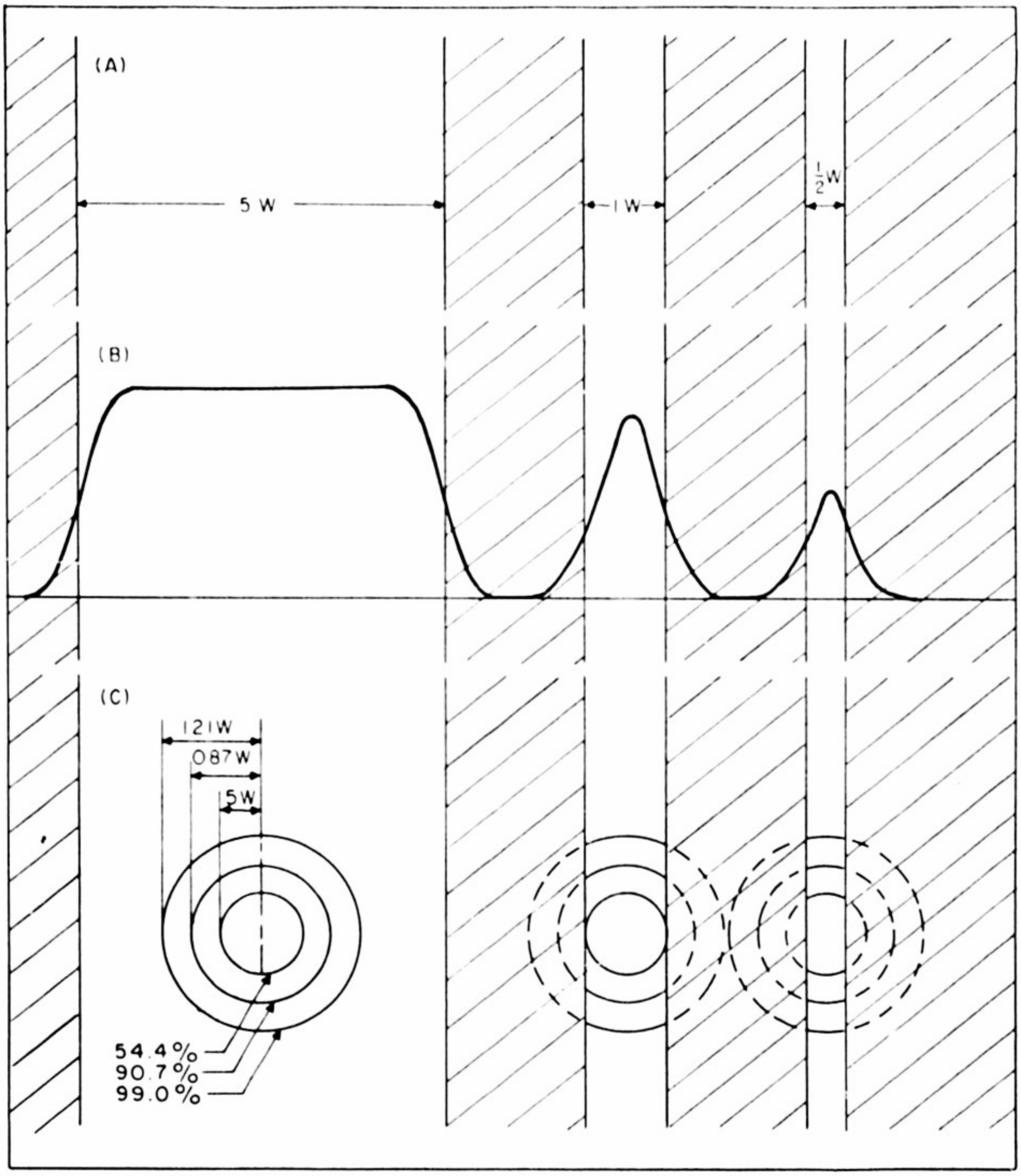

Figure 3 - Test Chart Function: (A) Test Grating; (B) Light Variation for Swept Beam; (C) Light Distribution about Centered Spot

the spot size, 21 percent of the light is obstructed by the opaque bars (see Figure 3C, center). In Figure 3C, right, the light contours permit the further reduction of maximum light associated with an aperture half the spot size to be visualized.

After the ratio is measured, Equation 5 may be used to solve for spot size. Alternatively, the graph of Figure 4 may be employed. The curve drawn here has a point of inflection at

$$\frac{w}{W} = \frac{\sqrt{\pi}}{2} = 0.88623 , \qquad (6)$$

as derived in Appendix A, Equation A-14. Consequently, in the region of this spot-size value, a small error in the reading of the light ratio has a minimum effect on the accuracy of the spot-size determination.

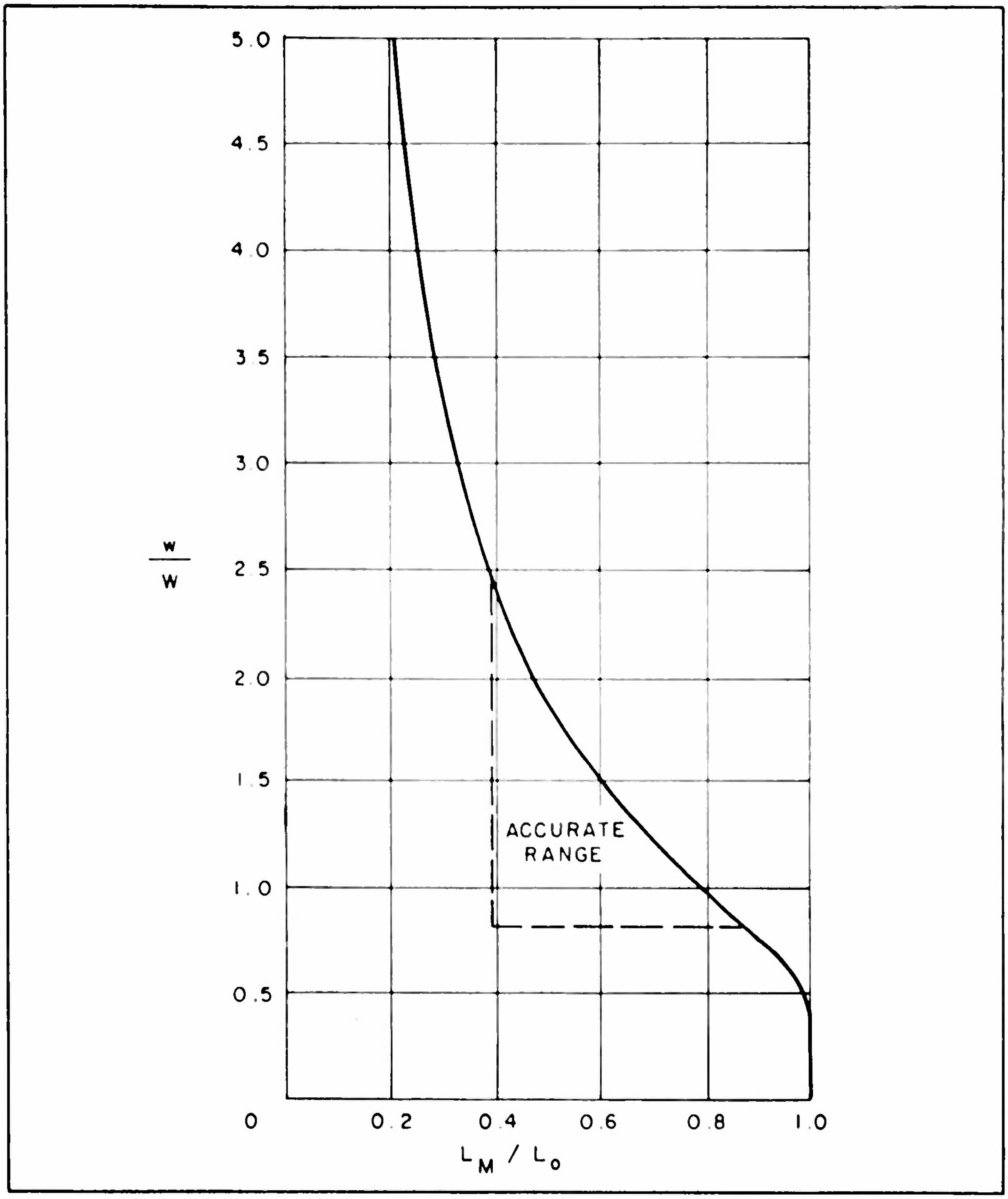

Figure 4 - Spot Size as Function of Light Ratio

SECTION IV - ERROR ANALYSIS

The expression for the maximum relative error is given in Equation A-24 as

$$\frac{\Delta w}{w} = \left(\frac{w}{W}\right) e^{\frac{\pi}{4}\left(\frac{w}{W}\right)^{-2}} \left[\frac{\Delta L_M}{L_o} - \frac{L_M}{L_o}\frac{\Delta L_o}{L_o}\right] + \frac{\Delta W}{W} . \qquad (7)$$

For a one-percent error in aperture width and in each recorded reading, the maximum relative error is given by Equation A-27 as

$$\frac{|\Delta w|}{w}(\max) \leq 0.02 \frac{w}{W} e^{\frac{\pi}{4}\left(\frac{w}{W}\right)^{-2}} + 0.01, \qquad (8)$$

and the rms value of the relative error (see Equation A-27) is

$$\frac{\Delta w}{w}(\mathrm{rms}) \leq \left[0.0002\left(\frac{w}{W}\right)^2 e^{\frac{\pi}{2}\left(\frac{w}{W}\right)^{-2}} + 0.0001\right]^{1/2} \qquad (9)$$

The error functions are plotted in Figures 5 and 6 for either one-or two-percent maximum error in each reading. From the graphs it is apparent that minimum relative error in the spot size measured is obtained when the spot size-to-aperture ratio is 1.25; and for a one-percent recording and reading error, the rms relative error is less than four percent over the range

$$0.08 < \frac{w}{W} < 2.4 \ . \qquad (10)$$

The maximum error is 6.5 percent over this range. Consequently, a grating designed for measurement of a wide range of spot sizes may have adjacent aperture widths in a 3:1 ratio.

It is also apparent from Figures 5 and 6 that, for a 2-percent recording and

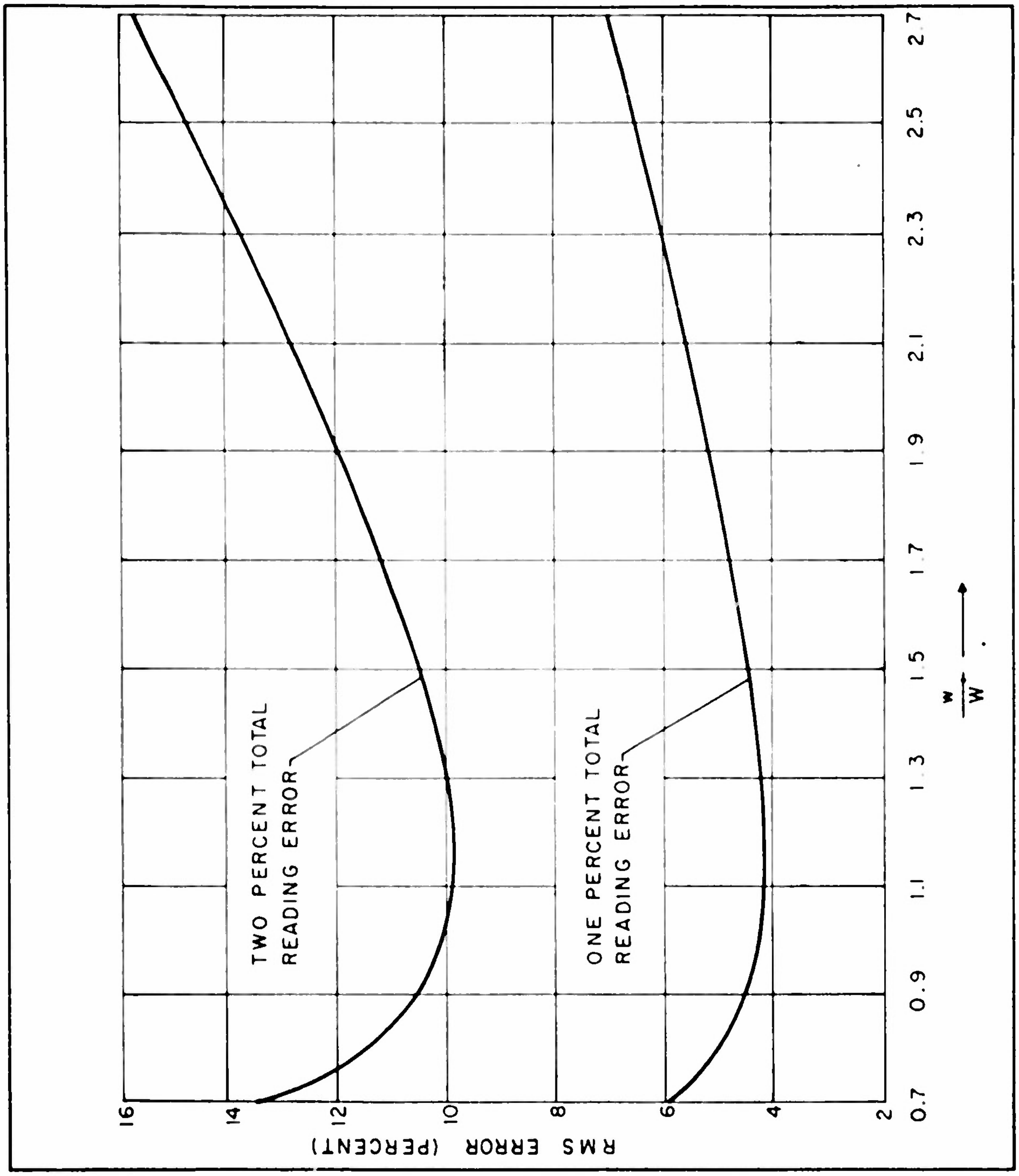

Figure 5 - RMS Value of Relative Spot-Size Error as Function of Spot-to-Aperture Ratio

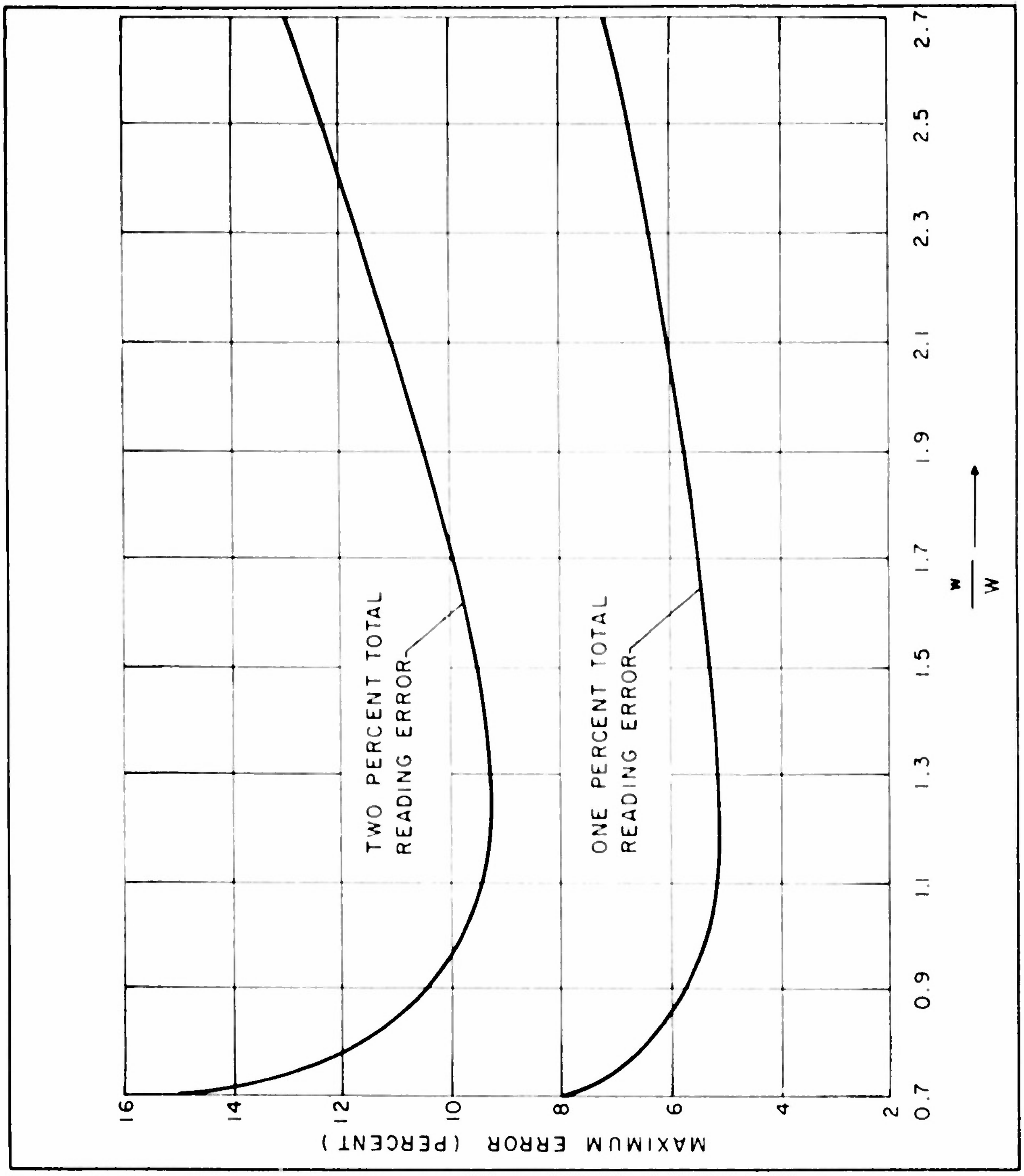

Figure 6 - Maximum Value of Relative Spot-Size Error as Function of Spot-to-Aperture Ratio

reading error, the rms relative error is less than 8 percent over approximately the same range as Equation 10, while the maximum error is 12 percent.

The accurate range region in Figure 4 corresponds to that specified in Equation 10.

The basic relation of this report is Equation 2, which was derived on the assumption that the distribution of the electrons in a beam is Gaussian, and that the light output of the phosphor (neglecting decay) is directly proportional to the current density. Consequently, the error equations will apply only when these assumptions are closely approximated.

LIST OF SYMBOLS

$H(x) \equiv \frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2} du$ = the probability integral

k = constant determined by definition of spot size for a Gaussian spot; for purposes of this report, it is equal to $\sqrt{\pi}/w$

L_o = reference level of light flux, measured when flying spot is focused on center of widest aperture on test chart

L_M = light flux incident upon photocathode when flying spot is focused on center of an aperture other than the reference aperture

L = light flux incident on photocathode after passing through an aperture on the slide

W = width of aperture

w = spot size of the cathode-ray tube

x = distance from center of spot to leading edge of aperture

APPENDIX A - DERIVATION OF ERROR EQUATIONS

The relative light output given by Equation 5 is

$$\frac{L_M}{L_o} = H\left(\frac{\sqrt{\pi}}{2} \frac{W}{w}\right) = \frac{2}{\sqrt{\pi}} \int_o^{\frac{\sqrt{\pi}}{2} \frac{W}{w}} e^{-u^2} du . \tag{A-1}$$

The derivatives of w/**W with** respect to the light output are found by defining z and y as

$$\left.\begin{aligned} z &= \frac{L_M}{L_o} \\ y &= \frac{2}{\sqrt{\pi}} \frac{w}{W} \\ f(y) &= \frac{2}{\sqrt{\pi}} \int_o^{\frac{1}{y}} e^{-u^2} du \end{aligned}\right\} , \tag{A-2}$$

so that Equation A-1 becomes

$$z = f(y), \tag{A-3}$$

and an implicit equation can be formed as

$$g(z, y) = z - f(y) = 0. \tag{A-4}$$

Then

$$\frac{dg}{dz} = 1 - \frac{\partial f(y)}{\partial y} \frac{dy}{dz} = 0, \tag{A-5}$$

and

$$\frac{d^2g}{dz^2} = 0 - \frac{\partial^2 f(y)}{\partial y^2}\left(\frac{dy}{dz}\right)^2 - \frac{\partial f(y)}{\partial y}\,\frac{d^2y}{dz^2} = 0. \qquad \text{(A-6)}$$

From Equations A-5 and A-6

$$\frac{dy}{dz} = \frac{1}{\frac{\partial f(y)}{\partial y}} = \frac{1}{\frac{df(y)}{dy}}$$

$$\frac{d^2y}{dz^2} = - \frac{\frac{\partial^2 f(y)}{\partial y^2}\left(\frac{dy}{dz}\right)^2}{\frac{\partial f(y)}{\partial y}} = - \frac{d^2f(y)}{dy^2}\left(\frac{dy}{dz}\right)^3 . \qquad \text{(A-7)}$$

The Leibnitz formula is used to find the derivative of f(y) for integrating under the integral sign; thus, from Equation A-2,

$$\frac{df(y)}{dy} = \frac{2}{\sqrt{\pi}}\, e^{-\left(\frac{1}{y}\right)^2}\left(-\frac{1}{y^2}\right) \equiv - \frac{2}{\sqrt{\pi}}\,\frac{e^{-(y^{-2})}}{y^2}. \qquad \text{(A-8)}$$

Then

$$\frac{d^2f}{dy^2} \equiv \frac{2}{\sqrt{\pi}}\, e^{-(y^{-2})}\left[\frac{2}{y^5} - \frac{2}{y^3}\right] \equiv \frac{4}{\sqrt{\pi}}\,\frac{e^{-\frac{1}{y^2}}}{y^5}\,(y^2 - 1). \qquad \text{(A-9)}$$

Substitution of Equations A-8 and A-9 into A-7 yields

$$\frac{dy}{dz} = - \frac{\sqrt{\pi}}{2}\, y^2 e^{y^{-2}}, \qquad \text{(A-10)}$$

and

$$\frac{d^2y}{dz^2} = \frac{4}{\sqrt{\pi}} \frac{e^{-(y^{-2})}}{y^5}(1 - y^2)\left(-\frac{\sqrt{\pi}}{2} y^2 e^{y^{-2}}\right)^3$$

$$= \frac{\pi}{2} y (1 - y^2) e^{2y^{-2}} . \qquad \text{(A-11)}$$

The original variables may be substituted for y and z to yield

$$\frac{\frac{2}{\sqrt{\pi}} d\left(\frac{w}{W}\right)}{d\left(\frac{L_M}{L_o}\right)} = -\frac{\sqrt{\pi}}{2} \times \frac{4}{\pi}\left(\frac{w}{W}\right)^2 e^{\frac{\pi}{4}\left(\frac{w}{W}\right)^{-2}} ,$$

or

$$\frac{d\left(\frac{w}{W}\right)}{d\left(\frac{L_M}{L_o}\right)} = -\left(\frac{w}{W}\right)^2 e^{\frac{\pi}{4}\left(\frac{w}{W}\right)^{-2}} , \qquad \text{(A-12)}$$

and

$$\frac{2}{\sqrt{\pi}} \frac{d^2\left(\frac{w}{W}\right)}{d\left(\frac{L_M}{L_o}\right)^2} = \frac{\pi}{2} \times \frac{2}{\sqrt{\pi}} \frac{w}{W}\left[1 - \frac{4}{\pi}\left(\frac{w}{W}\right)^2\right] e^{\frac{\pi}{4}\left(\frac{w}{W}\right)^{-2}} ,$$

or

$$\frac{d^2\left(\frac{w}{W}\right)}{d\left(\frac{L_M}{L_o}\right)^2} = \frac{\pi}{2} \frac{w}{W}\left[1 - \frac{4}{\pi}\left(\frac{w}{W}\right)^2\right] e^{\frac{\pi}{2}\left(\frac{w}{W}\right)^{-2}} . \qquad \text{(A-13)}$$

The last equation indicates a point of inflection at

$$\frac{w}{W} = \frac{\sqrt{\pi}}{2} = 0.88623 , \qquad \text{(A-14)}$$

where the value of the tangent is

$$\frac{d\left(\frac{w}{W}\right)}{d\left(\frac{L_M}{L_o}\right)} = \frac{\pi}{4} e = 2.135 \ . \tag{A-15}$$

When a small error in the measurement of the light ratio L_M/L is made, the approximate error in the spot-size ratio is obtained from Equation A-12 as

$$\Delta\left(\frac{w}{W}\right) = \left(\frac{w}{W}\right)^2 e^{\frac{\pi}{4}\times\left(\frac{w}{W}\right)^{-2}} \Delta\frac{L_M}{L_o} \ , \tag{A-16}$$

which is plotted in Figure A-1.

Since the quantities L_M and L are observed independently, the error in their ratio is

$$\Delta\left(\frac{L_M}{L_o}\right) = \frac{L_o \Delta L_M - L_M \Delta L_o}{L_o^2} \ . \tag{A-17}$$

Therefore, the maximum error in this ratio is

$$\Delta\left(\frac{L_M}{L_o}\right)(\max) = \frac{L_o\left|\Delta L_M\right| + L_M\left|\Delta L_o\right|}{L_o^2} \leq \frac{\left|\Delta L_M\right| + \left|\Delta L_o\right|}{L_o} , \tag{A-18}$$

and the rms error is

$$\Delta\left(\frac{L_M}{L_o}\right)(\mathrm{rms}) = \sqrt{\left(\frac{\Delta L_M}{L_o}\right)^2 + \left(\frac{L_M \Delta L_o}{L_o^2}\right)^2}$$

$$\leq \sqrt{\left(\frac{\Delta L_M}{L_o}\right)^2 + \left(\frac{\Delta L_o}{L_o}\right)^2} \ . \tag{A-19}$$

With the assumption that the error in each observation is either independent

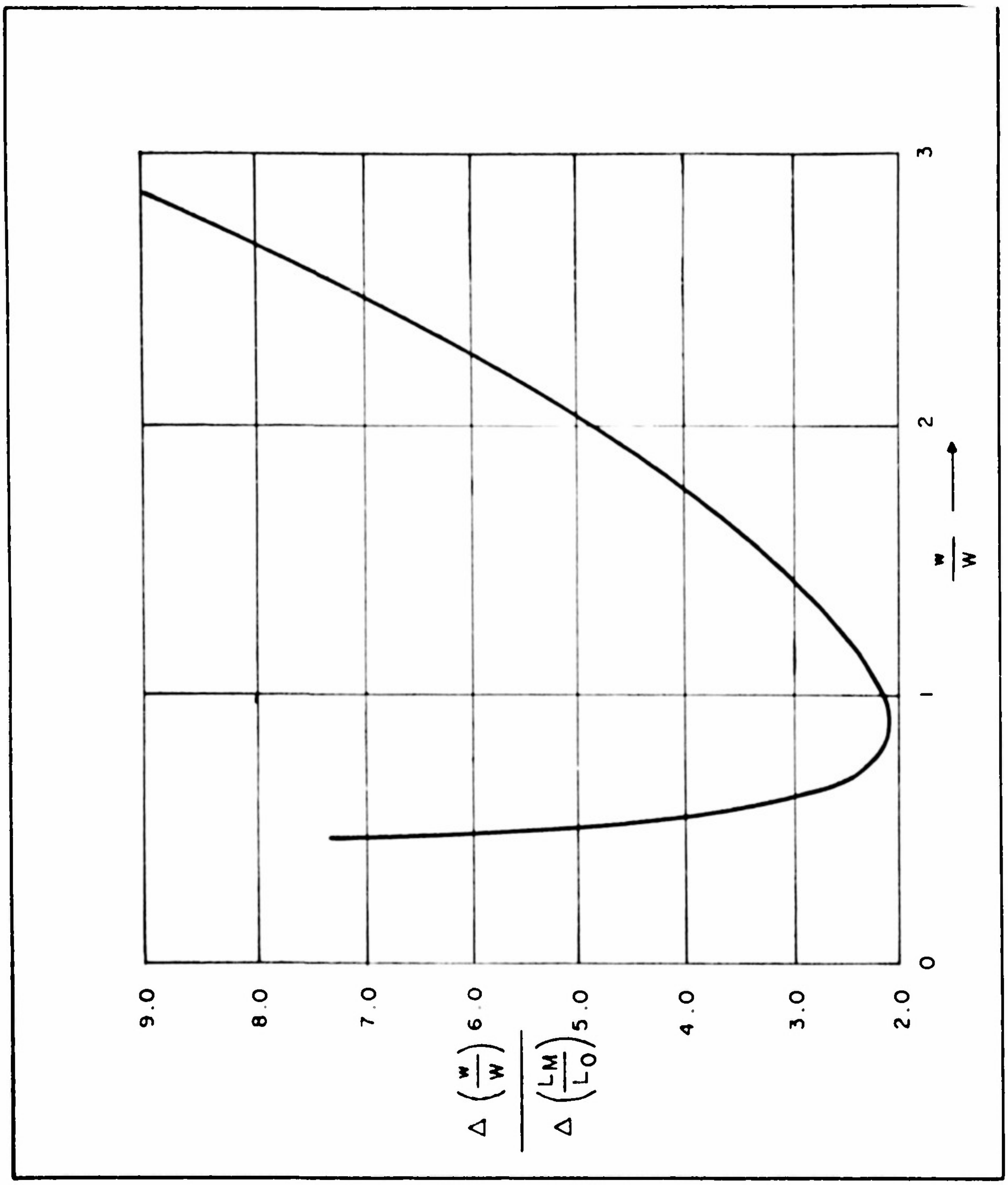

Figure A-1 - Graphic Representation of Equation A-16

of the amplitude, or else increases with amplitude, these relations become

$$\left.\begin{aligned} \Delta\left(\frac{L_M}{L_o}\right)(\max) &\leq 2\frac{|\Delta L_o|}{L_o} \\ \Delta\left(\frac{L_M}{L_o}\right)(\mathrm{rms}) &\leq \sqrt{2}\frac{|\Delta L_o|}{L_o} \end{aligned}\right\} . \qquad \text{(A-20)}$$

A reasonable value for the combined reading error and calibration error of a moving-stylus graphic recorder is

$$\frac{\Delta L_o}{L_o} = \pm\, 0.01 \ , \qquad \text{(A-21)}$$

so that

$$\left.\begin{aligned} \Delta\left(\frac{L_M}{L_o}\right)(\max) &= \pm\, 0.02 \\ \Delta\left(\frac{L_M}{L_o}\right)(\mathrm{rms}) &= 0.014 \end{aligned}\right\} . \qquad \text{(A-22)}$$

The spot size and the aperture width are independent quantities also, so that

$$\Delta\left(\frac{w}{W}\right) = \frac{\Delta w}{W} - \frac{w\Delta W}{W^2} \ ; \qquad \text{(A-23)}$$

therefore,

$$\frac{\Delta}{W} \quad \left(\frac{w}{W}\right) + \frac{w}{W}\frac{\Delta W}{W}$$

$$\left\{ [illegible] \frac{\pi}{4}\left(\frac{w}{W}\right)^{-2}\left[\frac{\Delta L_M}{L_o} - \frac{L_M}{L_o}\,\frac{\Delta L_o}{L_o}\right] + \frac{\Delta W}{W}\right\} . \qquad \text{(A-24)}$$

Then the maximum error is

$$\frac{|\Delta w|}{w}(\max) \leqq 2\frac{|\Delta L_o|}{L_o}\,\frac{w}{W}\,e^{\frac{\pi}{4}\left(\frac{w}{W}\right)^{-2}} + \frac{|\Delta W|}{W}, \tag{A-25}$$

while the rms error is given by

$$\frac{\Delta w}{w}(\text{rms}) \leqq \left[2\left(\frac{\Delta L_o}{L_o}\right)^2\left(\frac{w}{W}\right)^2 e^{\frac{\pi}{2}\left(\frac{w}{W}\right)^{-2}} + \left(\frac{\Delta W}{W}\right)^2\right]^{1/2}. \tag{A-26}$$

If the smallest aperture can be measured with one-percent accuracy, the above expressions, used with Equation A-21, become

$$\left.\begin{aligned} \frac{|\Delta w|}{w}(\max) &\leqq 0.02\,\frac{w}{W}\,e^{\frac{\pi}{4}\left(\frac{w}{W}\right)^{-2}} + 0.01 \\ \frac{\Delta w}{w}(\text{rms}) &\leqq \left[0.0002\left(\frac{w}{W}\right)^2 e^{\frac{\pi}{2}\left(\frac{w}{W}\right)^{-2}} + 0.0001\right]^{1/2} \end{aligned}\right\}. \tag{A-27}$$

If the output of the phototube is displayed on an oscilloscope of good design, the errors become

$$\left.\begin{aligned}
\frac{\Delta L_o}{L_o} &= \pm\, 0.02 \\
\Delta\left(\frac{L_M}{L_o}\right)(\max) &= \pm\, 0.04 \\
\Delta\left(\frac{L_M}{L_o}\right) &= 0.028 \\
\left|\frac{\Delta w}{w}\right|(\max) &\leqq 0.04\, \frac{w}{W}\, e^{\frac{\pi}{4}\left(\frac{w}{W}\right)^{-2}} + 0.01 \\
\frac{\Delta w}{w}(\mathrm{rms}) &\leqq \left[0.0008\left(\frac{w}{W}\right)^2 e^{\frac{\pi}{2}\left(\frac{w}{W}\right)^{-2}} + 0.0001\right]^{1/2}
\end{aligned}\right\} \quad . \tag{A-28}$$